

INTERTWININGS OF BETA-DYSON BROWNIAN MOTIONS OF DIFFERENT DIMENSIONS

KAVITA RAMANAN AND MYKHAYLO SHKOLNIKOV

ABSTRACT. We show that for all positive β the semigroups of β -Dyson Brownian motions of different dimensions are intertwined. The proof relates β -Dyson Brownian motions directly to Jack symmetric polynomials and omits an approximation of the former by discrete space Markov chains, thereby disposing of the technical assumption $\beta \geq 1$ in [GS]. The corresponding results for β -Dyson Ornstein-Uhlenbeck processes are also presented.

1. INTRODUCTION

For a fixed $\beta > 0$ consider the system of stochastic differential equations

$$(1.1) \quad dX_i^{(N)}(t) = \frac{\beta}{2} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \frac{1}{X_i^{(N)}(t) - X_j^{(N)}(t)} dt + dB_i^{(N)}(t), \quad i = 1, 2, \dots, N,$$

with initial condition satisfying $X_1^{(N)}(0) \leq X_2^{(N)}(0) \leq \dots \leq X_N^{(N)}(0)$ and where $B^{(N)} = (B_1^{(N)}, B_2^{(N)}, \dots, B_N^{(N)})$ is a standard N -dimensional Brownian motion. For $\beta = 1, 2, 4$ it has been shown by Dyson [Dy] that the equation (1.1) describes the evolution of eigenvalues of random symmetric matrices whose entries follow real, complex, and quaternion Brownian motions, respectively. More generally, one can make sense of the unique strong solution to (1.1) for all $\beta > 0$ (see [CL1, Theorem 3.1] and also the alternative proof in [AGZ, Proposition 4.3.5] for $\beta \geq 1$). We refer to the latter as the N -dimensional β -Dyson Brownian motion and write $P^{(N)}(t)$, $t \geq 0$, for the associated Markov transition operators, emphasizing the dependence on dimension and omitting the explicit dependence on β .

More recently, Dyson Brownian motions have been used to establish a variety of universality conjectures in random matrix theory, such as universality of the local eigenvalue statistics in the bulk and at the edge of the spectrum of symmetric Wigner matrices (see [ESY] and [BEY], respectively), universality of the local statistics in the bulk and at the edge of non-critical beta ensembles with smooth external potentials (see [BEY]) and the universality of the bulk statistics and the distribution of the second largest eigenvalue of dense Erdős-Rényi random graphs (see [EKYY1], [EKYY2]) among others.

Concurrently, it was observed in [Ba] that for $\beta = 2$ and densely packed initial conditions $X_1^{(N)}(0) = X_2^{(N)}(0) = \dots = X_N^{(N)}(0)$, the largest coordinate process $X_N^{(N)}$ in an N -dimensional Dyson Brownian motion has the same distribution as the rightmost

2010 *Mathematics Subject Classification.* 60H10, 33D52, 82C22.

Key words and phrases. Dixon-Anderson conditional probability density, Dyson Brownian motions, Dyson Ornstein-Uhlenbeck processes, Gaussian random matrix ensembles, intertwining, Jack symmetric polynomials.

The first author, K. Ramanan, is partially supported by NSF grant DMS-1407504, and the second author, M. Shkolnikov, is partially supported by NSF grant DMS-1506290.

$$\begin{array}{ccc}
X^{(k+1)}(0) & \xrightarrow{P^{(k+1)}(t)} & X^{(k+1)}(t) \\
\downarrow L^{(k)} & & \downarrow L^{(k)} \\
X^{(k)}(0) & \xrightarrow{P^{(k)}(t)} & X^{(k)}(t)
\end{array}$$

FIGURE 1. Intertwining of $P^{(k)}$ and $P^{(k+1)}$.

particle in the Brownian totally asymmetric simple exclusion process (TASEP) started from the same densely packed initial condition. This connection reveals that the fixed time fluctuations of the rightmost particle in the Brownian TASEP are governed by the same distribution as those of the largest eigenvalue of a Hermitian random matrix, namely the Tracy-Widom distribution TW_2 .

The seminal paper [Wa] explains the relation between the $(\beta = 2)$ -Dyson Brownian motion and the Brownian TASEP by constructing a process $(X_i^{(k)}, 1 \leq i \leq k \leq N)$ taking values in the Gelfand-Tseitin cone

(1.2)

$$\mathcal{G}^{(N)} := \left\{ (x_i^{(k)} : 1 \leq i \leq k \leq N) \in \mathbb{R}^{N(N+1)/2} : x_i^k \leq x_{i+1}^{k-1} \leq x_{i+1}^k, 1 \leq i \leq k \leq N-1 \right\}$$

and starting from $0 \in \mathcal{G}^{(N)}$ such that each “level” $X^{(k)} := (X_1^{(k)}, X_2^{(k)}, \dots, X_k^{(k)})$ performs a k -dimensional $(\beta = 2)$ -Dyson Brownian motion, whereas the “diagonal” $(X_1^{(1)}, X_2^{(2)}, \dots, X_N^{(N)})$ forms the Brownian TASEP with N particles. We refer to $(X_i^{(k)}, 1 \leq i \leq k \leq N)$ as the $(\beta = 2)$ -multilevel Dyson Brownian motion. The construction in [Wa] relies on the fact that for $\beta = 2$ and any $k \geq 1$ the semigroups $P^{(k)}(t)$, $t \geq 0$, and $P^{(k+1)}(t)$, $t \geq 0$, are *intertwined*. More specifically, there exists a stochastic kernel $L^{(k)}(x^{(k+1)}, dx^{(k)})$ such that

$$(1.3) \quad L^{(k)} P^{(k)}(t) = P^{(k+1)}(t) L^{(k)}, \quad t \geq 0.$$

Figure 1 provides an illustration of the intertwining relationship (1.3).

By now intertwining have been used to construct a number of important “multi-level” processes. For discrete space Markov chains the procedure is based on the coupling of Diaconis and Fill [DF] and has been applied to construct multilevel Markov chains describing the growth of two-dimensional random surfaces, see [BC], [BF2], [BK], [GS] and the references therein. Such random surfaces are of great interest as members of the $(2+1)$ -dimensional Kardar-Parisi-Zhang (KPZ) universality class and yield insights into solutions of the KPZ stochastic partial differential equation.

The construction of multilevel processes with continuous state space presents additional technical challenges, because the transition operators of the underlying Markov processes are usually not available in closed form (the situation of [Wa] being a notable exception). For this reason, there are only three known examples in the continuous space setting: the two-dimensional Whittaker growth model of [OC], the $\beta = 2$ multilevel Dyson Brownian motion of [Wa] and the $\beta > 2$ multilevel Dyson Brownian motions of [GS]. The article [OC] is the only one where the intertwining relationship

behind the multilevel continuous space process is established directly and in the absence of explicit formulas for the transition operators. In contrast, [Wa] relies on such explicit formulas and the proof in [GS] is based on an approximation by discrete space Markov chains.

In this article we make progress towards the construction of multilevel Dyson Brownian motions for all positive β by extending the intertwining relationship (1.3) to all $\beta > 0$. In contrast with [GS], our proof works directly with β -Dyson Brownian motions rather than their discrete space approximations. In the process, we provide formulas for how β -Dyson Brownian motion generators and semigroups act on Jack symmetric polynomials.

To state our main result we need the following set of notations. We define the chambers

$$(1.4) \quad \mathcal{W}^{(k+1)} := \{x^{(k+1)} \in \mathbb{R}^{k+1} : x_1^{(k+1)} \leq x_2^{(k+1)} \leq \dots \leq x_{k+1}^{(k+1)}\},$$

$$(1.5) \quad \mathcal{W}^{(k)}(x^{(k+1)}) := \{x^{(k)} \in \mathbb{R}^k : x_1^{(k+1)} \leq x_1^{(k)} \leq x_2^{(k+1)} \leq \dots \leq x_k^{(k)} \leq x_{k+1}^{(k+1)}\},$$

for $x^{(k+1)} \in \mathcal{W}^{(k+1)}$. In addition, we fix $\beta > 0$ and for each $x^{(k+1)} \in \mathcal{W}^{(k+1)}$ we write

$$(1.6) \quad \Lambda^{(k)}(x^{(k+1)}, x^{(k)}) := \frac{\Gamma(\beta(k+1)/2)}{\Gamma(\beta/2)^{k+1}} \prod_{1 \leq i < j \leq k+1} (x_j^{(k+1)} - x_i^{(k+1)})^{1-\beta} \prod_{1 \leq i < j \leq k} (x_j^{(k)} - x_i^{(k)}) \prod_{i=1}^k \prod_{j=1}^{k+1} |x_i^{(k)} - x_j^{(k+1)}|^{\beta/2-1}$$

for the Dixon-Anderson conditional probability density of $x^{(k)}$ on $\mathcal{W}^{(k)}(x^{(k+1)})$ given $x^{(k+1)}$, where $\Gamma(\cdot)$ stands for the gamma function. The density $\Lambda^{(k)}(x^{(k+1)}, \cdot)$ has been introduced independently in [Di] and [An] and for $\beta = 1, 2, 4$ describes the conditional distribution of the eigenvalues of the $k \times k$ corner in a $(k+1) \times (k+1)$ random matrix from the Gaussian orthogonal, unitary, symplectic ensemble, respectively (see also [Fo1] for a more detailed discussion and generalizations).

Our main result reads as follows.

Theorem 1.1. *Let $\beta > 0$. Then, with the Dixon-Anderson conditional probability density of (1.6), the stochastic kernel $L^{(k)}(x^{(k+1)}, dx^{(k)}) := \Lambda^{(k)}(x^{(k+1)}, x^{(k)}) dx^{(k)}$ intertwines the semigroups $P^{(k)}$ and $P^{(k+1)}$ of the k -dimensional and the $(k+1)$ -dimensional β -Dyson Brownian motions, respectively. In other words, one has the following equalities of probability measures:*

$$(1.7) \quad L^{(k)} P^{(k)}(t) = P^{(k+1)}(t) L^{(k)}, \quad t \geq 0.$$

Remark 1.2. An iteration of (1.7) shows that for all $1 \leq k < n$, the following equality holds

$$(1.8) \quad \left(\prod_{m=k}^{n-1} L^{(m)} \right) P^{(k)}(t) = P^{(n)}(t) \left(\prod_{m=k}^{n-1} L^{(m)} \right), \quad t \geq 0.$$

Remark 1.3. The general results on intertwining of diffusion processes in [PS] (see, in particular, [PS, Theorem 5]) suggest that one should be able to realize the intertwining relationships (1.7) for $k = 1, 2, \dots, N-1$ by a coupling of β -Dyson Brownian motions

of dimensions $1, 2, \dots, N$ to a multilevel process on $\mathcal{G}^{(N)}$. Moreover, an iterative application of the formulas in [PS, Theorem 5] leads one to conjecture that the generator of such a coupling should be given by

$$(1.9) \quad \frac{1}{2} \sum_{1 \leq i \leq k \leq N} \partial_{x_i^{(k)}}^2 - \sum_{1 \leq i \leq k \leq N} \sum_{j \neq i} \frac{\beta/2 - 1}{x_i^{(k)} - x_j^{(k)}} \partial_{x_i^{(k)}} + \sum_{1 \leq i \leq k+1 \leq N} \sum_{j=1}^k \frac{\beta/2 - 1}{x_i^{(k+1)} - x_j^{(k)}} \partial_{x_i^{(k+1)}},$$

endowed with the reflecting boundary conditions described in [Wa]. Processes of this type have indeed been constructed in [Wa] for $\beta = 2$ and in [GS] for $\beta > 2$ and one should view them as $\mathcal{G}^{(N)}$ -valued analogues of Bessel processes of dimensions $d = 1$ and $d > 1$, respectively. This analogy suggests that for $\beta < 2$ a process with generator (1.9) should no longer be a semimartingale. The construction of such a process appears challenging: the non-reversibility of the process in conjunction with the reflecting boundary conditions rule out non-symmetric Dirichlet form techniques (note that the process constructed in [MR, Exercise II.2.14, Theorem IV.3.5] is absorbed at the boundary); on the other hand, the singularity of the coefficients together with the inapplicability of localization techniques as in [GS] prevent one from directly building the coupling from known processes.

We conclude the introduction by presenting the version of Theorem 1.1 for β -Dyson Ornstein-Uhlenbeck processes. Fix a $\beta > 0$ and consider the system of stochastic differential equations

$$(1.10) \quad dY_i^{(N)}(t) = \frac{\beta}{2} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \frac{1}{Y_i^{(N)}(t) - Y_j^{(N)}(t)} dt - \frac{Y_i^{(N)}(t)}{2} dt + dB_i^{(N)}(t),$$

$i = 1, 2, \dots, N$, with initial condition $Y_1^{(N)}(0) \leq Y_2^{(N)}(0) \leq \dots \leq Y_N^{(N)}(0)$ and where $B^{(N)} = (B_1^{(N)}, B_2^{(N)}, \dots, B_N^{(N)})$ is a standard N -dimensional Brownian motion as before. We call the unique strong solution of (1.10) (see [CL1, Theorem 3.1]) the N -dimensional β -Dyson Ornstein-Uhlenbeck process and denote the associated Markov transition operators by $Q^{(N)}(t)$, $t \geq 0$.

Proposition 1.4. *Let $\beta > 0$. Then, with the notations of Theorem 1.1 and the previous paragraph, one has the intertwining relationships*

$$(1.11) \quad L^{(k)} Q^{(k)}(t) = Q^{(k+1)}(t) L^{(k)}, \quad t \geq 0,$$

for all $k \geq 1$.

Remark 1.5. As in Remark 1.2, (1.8) holds with $P^{(k)}$ and $P^{(n)}$ replaced by $Q^{(k)}$ and $Q^{(n)}$, respectively.

The rest of the paper is structured as follows. In Section 2 we present some facts about Jack symmetric polynomials that are needed in the proof of Theorem 1.1. Section 3 is then devoted to the proof of Theorem 1.1. In Section 4 we present a simpler proof for the case $\beta = 1$ based on the random matrix interpretation of $\beta = 1$ Dyson Brownian motions. Finally, in Section 5 we give the proof of Proposition 1.4.

2. PRELIMINARIES ON JACK SYMMETRIC POLYNOMIALS

We start with the definition of Jack symmetric polynomials following [BF1, Section 2], but replacing the parameter α there by $\theta := 1/\alpha$ as in [OO].

Definition 2.1. Let $\theta := \beta/2$ and consider the differential operator

$$(2.1) \quad \sum_{i=1}^k z_i^2 \frac{\partial^2}{\partial z_i^2} + 2\theta \sum_{i=1}^k \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \frac{z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i}$$

acting on symmetric polynomials in k variables. It is known (see [St, Theorem 3.1]) that the eigenfunctions of this operator can be indexed by non-decreasing positive integer sequences $\kappa = (\kappa_i, i \leq \ell)$, with $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_\ell > 0$ and $\ell \leq k$, such that the lexicographically leading monomial with a non-zero coefficient in the eigenfunction $J_\kappa(\cdot; \theta)$ is $z_1^{\kappa_1} z_2^{\kappa_2} \dots z_\ell^{\kappa_\ell}$ and $J_\kappa(\cdot; \theta)$ is normalized according to

$$(2.2) \quad J_\kappa(1_k; \theta) = \prod_{i=1}^l \prod_{j=1}^{\kappa_i} (k+1-i+(j-1)\theta) = \theta^{-|\kappa|} \prod_{i=1}^l \frac{\Gamma((k+1-i)\theta + \kappa_i)}{\Gamma((k+1-i)\theta)}.$$

Here, 1_k is the k -dimensional vector whose components are all equal to 1 and $|\kappa| := \sum_{i=1}^l \kappa_i$. The eigenfunction $J_\kappa(\cdot; \theta)$ is called the Jack symmetric polynomial in k variables with parameters κ, θ .

Remark 2.2. Jack symmetric polynomials can be also defined as the eigenfunctions of the Sekiguchi differential operators

$$(2.3) \quad \frac{1}{\prod_{1 \leq i < j \leq k} (z_i - z_j)} \det \left[z_i^{k-j} \left(z_i \frac{\partial}{\partial z_i} + (k-j)\theta + u \right) \right]_{1 \leq i, j \leq k}$$

in the space of symmetric polynomials in k variables (see, e.g., [OO, Section 1] and the references therein). The equivalence of the two definitions follows from the fact that the operator (2.1) can be recovered by an affine transformation of the operator multiplying u^{k-2} in the expansion of (2.3) in powers of the auxiliary variable u (see [Ma, Example VI.3.3]).

We proceed to the definition of the generalized binomial coefficients associated with shifts of Jack symmetric polynomials (see [OO, remark on p.73] for more details).

Definition 2.3. The coefficients $\binom{\kappa}{\rho}_\theta$ in the expansion

$$(2.4) \quad \frac{J_\kappa(1_k + z; \theta)}{J_\kappa(1_k; \theta)} = \sum_{m=0}^{|\kappa|} \sum_{|\rho|=m} \binom{\kappa}{\rho}_\theta \frac{J_\rho(z; \theta)}{J_\rho(1_k; \theta)}$$

are referred to as the generalized binomial coefficients.

Remark 2.4. Note that the definition of $\binom{\kappa}{\rho}_\theta$ does not depend on the particular normalization of the Jack symmetric polynomials.

Next, we state three identities for Jack symmetric polynomials from [BF1, equations (2.13a), (2.13b), (2.13d)] that will be employed in the proofs of Theorem 1.1 and Proposition 1.4.

Proposition 2.5. *The differential operators*

$$\begin{aligned}
 \mathcal{B}_1 &:= \sum_{i=1}^k \frac{\partial}{\partial z_i}, \\
 \mathcal{B}_2 &:= \frac{1}{2} \sum_{i=1}^k z_i \frac{\partial^2}{\partial z_i^2} + \theta \sum_{i=1}^k \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \frac{z_i}{z_i - z_j} \frac{\partial}{\partial z_i}, \\
 \mathcal{B}_3 &:= \sum_{i=1}^k z_i \frac{\partial}{\partial z_i},
 \end{aligned}
 \tag{2.5}$$

act on Jack symmetric polynomials in k variables with parameter θ as follows:

$$\mathcal{B}_1 J_\kappa(z; \theta) = J_\kappa(1_k; \theta) \sum_{i=1}^l \binom{\kappa}{\kappa_{(i)}}_\theta \frac{J_{\kappa_{(i)}}(z; \theta)}{J_{\kappa_{(i)}}(1_k; \theta)},
 \tag{2.6}$$

$$\mathcal{B}_2 J_\kappa(z; \theta) = J_\kappa(1_k; \theta) \sum_{i=1}^l \binom{\kappa}{\kappa_{(i)}}_\theta (\kappa_i - 1 + (k - i) \theta) \frac{J_{\kappa_{(i)}}(z; \theta)}{J_{\kappa_{(i)}}(1_k; \theta)},
 \tag{2.7}$$

$$\mathcal{B}_3 J_\kappa(z; \theta) = |\kappa| J_\kappa(z; \theta).
 \tag{2.8}$$

Here $\kappa_{(i)}$ is the sequence obtained from κ by replacing κ_i by $\kappa_i - 1$ unless $i = l$ and $\kappa_l = 1$ in which case we drop κ_l from κ . We have also set $\binom{\kappa}{\kappa_{(i)}}_\theta = 0$ whenever $\kappa_{(i)}$ is no longer a non-decreasing positive sequence.

The last ingredient we need is a formula for the action of the Dixon-Anderson conditional probability density of (1.6) on the Jack symmetric polynomials in k variables. A proof of this formula can be found in [OO, Section 6] (note that the particular normalization of the Jack symmetric polynomials is irrelevant here).

Proposition 2.6. *With $\Lambda^{(k)}$ of (1.6) and $\theta = \beta/2$ one has for every $x^{(k+1)} \in \mathcal{W}^{(k+1)}$ and sequence κ ,*

$$\begin{aligned}
 &\int_{\mathcal{W}^k(x^{(k+1)})} \Lambda^{(k)}(x^{(k+1)}, x^{(k)}) J_\kappa(x^{(k)}; \theta) dx^{(k)} \\
 &= J_\kappa(x^{(k+1)}; \theta) \frac{\Gamma((k+1)\theta)}{\Gamma(\theta)} \prod_{i=1}^k \frac{\Gamma((k+1-i)\theta + \kappa_i)}{\Gamma((k+2-i)\theta + \kappa_i)},
 \end{aligned}
 \tag{2.9}$$

where we use the convention $\kappa_i = 0$ whenever i exceeds the length of κ .

3. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. For $\beta = 2$ the result of Theorem 1.1 has been established in [Wa, Section 3] by a direct computation exploiting the explicit formulas for the transition densities of the $(\beta = 2)$ -Dyson Brownian motions. Later, the statement of Theorem 1.1 was shown to hold for $\beta \geq 1$ in [GS, Proposition 1.3]. The proof there relies on a construction of a sequence of continuous time Markov chains realizing a discrete version of the intertwining (1.7) in the sense of [DF]. This sequence of Markov chains is then shown to be tight, with every limit point realizing the intertwining (1.7). As announced in the introduction, our proof is direct and uses only the properties of Jack symmetric polynomials stated in Section 2.

Proof of Theorem 1.1. Step 1. To obtain the theorem we are going to show that for a measure-determining class of test functions, the integrals of these functions

with respect to the probability measures on both sides of (1.7) are the same. We start by proving the corresponding identity with the Jack symmetric polynomials in k variables as test functions and the semigroups $P^{(k)}(t)$, $t \geq 0$, and $P^{(k+1)}(t)$, $t \geq 0$, in (1.7) replaced by their respective generators. More specifically, with

$$(3.1) \quad \mathcal{A}^{(k)} := \frac{1}{2} \sum_{i=1}^k \frac{\partial^2}{\partial z_i^2} + \theta \sum_{i=1}^k \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \frac{1}{z_i - z_j} \frac{\partial}{\partial z_i},$$

our first aim is to show

$$(3.2) \quad L^{(k)} \mathcal{A}^{(k)} J_\kappa(\cdot; \theta) = \mathcal{A}^{(k+1)} L^{(k+1)} J_\kappa(\cdot; \theta).$$

The key observation in the evaluation of $\mathcal{A}^{(k)} J_\kappa(\cdot; \theta)$ is that

$$(3.3) \quad \mathcal{A}^{(k)} = \mathcal{B}_1 \mathcal{B}_2 - \mathcal{B}_2 \mathcal{B}_1$$

with the operators $\mathcal{B}_1, \mathcal{B}_2$ of Proposition 2.5. This is indicated in [BF1, comment after equation (2.13e)] without proof and can be seen as follows:

$$\begin{aligned} & \mathcal{B}_1 \mathcal{B}_2 - \mathcal{B}_2 \mathcal{B}_1 \\ &= \frac{1}{2} \sum_{m,i=1}^k \left(\frac{\partial}{\partial z_m} z_i \frac{\partial^2}{\partial z_i^2} - z_i \frac{\partial^2}{\partial z_i^2} \frac{\partial}{\partial z_m} \right) + \theta \sum_{\substack{1 \leq m,i,j \leq k \\ j \neq i}} \left(\frac{\partial}{\partial z_m} \frac{z_i}{z_i - z_j} \frac{\partial}{\partial z_i} - \frac{z_i}{z_i - z_j} \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_m} \right) \\ &= \frac{1}{2} \sum_{m,i=1}^k \delta_{mi} \frac{\partial^2}{\partial z_i^2} + \theta \sum_{\substack{1 \leq m,i,j \leq k \\ j \neq i}} \frac{(z_i - z_j) \delta_{mi} - z_i (\delta_{mi} - \delta_{mj})}{(z_i - z_j)^2} \frac{\partial}{\partial z_i} = \mathcal{A}^{(k)}, \end{aligned}$$

where δ_{mi}, δ_{mj} stand for Kronecker deltas.

Combining (3.3) with the identities of Proposition 2.5 we find that $\mathcal{A}^{(k)} J_\kappa(x^{(k)}; \theta)$ is equal to

$$\begin{aligned} (3.4) \quad & J_\kappa(1_k; \theta) \sum_{i=1}^l \left[\left(\begin{matrix} \kappa \\ \kappa_{(i)} \end{matrix} \right)_\theta (\kappa_i - 1 + (k - i) \theta) \frac{\mathcal{B}_1 J_{\kappa_{(i)}}(x^{(k)}; \theta)}{J_{\kappa_{(i)}}(1_k; \theta)} \right. \\ & \quad \left. - J_\kappa(1_k; \theta) \sum_{i=1}^l \left(\begin{matrix} \kappa \\ \kappa_{(i)} \end{matrix} \right)_\theta \frac{\mathcal{B}_2 J_{\kappa_{(i)}}(x^{(k)}; \theta)}{J_{\kappa_{(i)}}(1_k; \theta)} \right] \\ &= J_\kappa(1_k; \theta) \sum_{i=1}^l \left(\begin{matrix} \kappa \\ \kappa_{(i)} \end{matrix} \right)_\theta (\kappa_i - 1 + (k - i) \theta) \sum_{j=1}^{l_i} \left(\begin{matrix} \kappa_{(i)} \\ (\kappa_{(i)})_{(j)} \end{matrix} \right)_\theta \frac{J_{(\kappa_{(i)})_{(j)}}(x^{(k)}; \theta)}{J_{(\kappa_{(i)})_{(j)}}(1_k; \theta)} \\ & \quad - J_\kappa(1_k; \theta) \sum_{i=1}^l \left(\begin{matrix} \kappa \\ \kappa_{(i)} \end{matrix} \right)_\theta \sum_{j=1}^{l_i} \left(\begin{matrix} \kappa_{(i)} \\ (\kappa_{(i)})_{(j)} \end{matrix} \right)_\theta ((\kappa_{(i)})_j - 1 + (k - j) \theta) \frac{J_{(\kappa_{(i)})_{(j)}}(x^{(k)}; \theta)}{J_{(\kappa_{(i)})_{(j)}}(1_k; \theta)} \\ &= J_\kappa(1_k; \theta) \sum_{i=1}^l \sum_{j=1}^{l_i} \left(\begin{matrix} \kappa \\ \kappa_{(i)} \end{matrix} \right)_\theta \left(\begin{matrix} \kappa_{(i)} \\ (\kappa_{(i)})_{(j)} \end{matrix} \right)_\theta (\kappa_i - (\kappa_{(i)})_j + (j - i) \theta) \frac{J_{(\kappa_{(i)})_{(j)}}(x^{(k)}; \theta)}{J_{(\kappa_{(i)})_{(j)}}(1_k; \theta)}, \end{aligned}$$

where l and l_i denote the lengths of the sequences κ and $\kappa_{(i)}$, respectively. Now, Proposition 2.6 yields

$$(3.5) \quad \begin{aligned} & \int_{\mathcal{W}^k(x^{(k+1)})} \Lambda(x^{(k+1)}, x^{(k)}) \mathcal{A}^{(k)} J_{\kappa}(x^{(k)}; \theta) dx^{(k)} \\ &= J_{\kappa}(1_k; \theta) \sum_{i=1}^l \sum_{j=1}^{l_i} \left[\binom{\kappa}{\kappa_{(i)}}_{\theta} \binom{\kappa_{(i)}}{(\kappa_{(i)})_{(j)}}_{\theta} (\kappa_i - (\kappa_{(i)})_j + (j-i)\theta) \frac{J_{(\kappa_{(i)})_{(j)}}(x^{(k+1)}; \theta)}{J_{(\kappa_{(i)})_{(j)}}(1_k; \theta)} \right. \\ & \quad \left. \cdot \frac{\Gamma((k+1)\theta)}{\Gamma(\theta)} \prod_{m=1}^k \frac{\Gamma((k+1-m)\theta + ((\kappa_{(i)})_{(j)})_m)}{\Gamma((k+2-m)\theta + ((\kappa_{(i)})_{(j)})_m)} \right], \end{aligned}$$

with the convention $((\kappa_{(i)})_{(j)})_i = 0$ whenever i exceeds the length of $(\kappa_{(i)})_{(j)}$. On the other hand, applying $\mathcal{A}^{(k+1)}$ to both sides of (2.9) and using (3.4), with k replaced by $k+1$, we obtain

$$(3.6) \quad \begin{aligned} & \mathcal{A}^{(k+1)} \int_{\mathcal{W}^k(x^{(k+1)})} \Lambda^{(k)}(x^{(k+1)}, x^{(k)}) J_{\kappa}(x^{(k)}; \theta) dx^{(k)} \\ &= J_{\kappa}(1_{k+1}; \theta) \sum_{i=1}^l \sum_{j=1}^{l_i} \left[\binom{\kappa}{\kappa_{(i)}}_{\theta} \binom{\kappa_{(i)}}{(\kappa_{(i)})_{(j)}}_{\theta} \cdot (\kappa_i - (\kappa_{(i)})_j + (j-i)\theta) \frac{J_{(\kappa_{(i)})_{(j)}}(x^{(k+1)}; \theta)}{J_{(\kappa_{(i)})_{(j)}}(1_{k+1}; \theta)} \right. \\ & \quad \left. \cdot \frac{\Gamma((k+1)\theta)}{\Gamma(\theta)} \prod_{m=1}^k \frac{\Gamma((k+1-m)\theta + \kappa_m)}{\Gamma((k+2-m)\theta + \kappa_m)} \right]. \end{aligned}$$

We claim that the right-hand sides of (3.5) and (3.6) are identical. Indeed, an inspection of the coefficients of $J_{(\kappa_{(i)})_{(j)}}(x^{(k+1)}; \theta)$ in both expressions shows that it suffices to check that

$$(3.7) \quad \begin{aligned} & \frac{J_{\kappa}(1_k; \theta)}{J_{\kappa}(1_{k+1}; \theta)} \prod_{m=1}^k \frac{\Gamma((k+2-m)\theta + \kappa_m)}{\Gamma((k+1-m)\theta + \kappa_m)} \\ &= \frac{J_{(\kappa_{(i)})_{(j)}}(1_k; \theta)}{J_{(\kappa_{(i)})_{(j)}}(1_{k+1}; \theta)} \prod_{m=1}^k \frac{\Gamma((k+2-m)\theta + ((\kappa_{(i)})_{(j)})_m)}{\Gamma((k+1-m)\theta + ((\kappa_{(i)})_{(j)})_m)}. \end{aligned}$$

However, this holds because, using the second formula in (2.2), both sides of (3.7) can be shown to be equal to

$$\prod_{m=1}^k \frac{\Gamma((k+2-m)\theta)}{\Gamma((k+1-m)\theta)} = \frac{\Gamma((k+1)\theta)}{\Gamma(\theta)}.$$

This finishes the proof of (3.2).

Step 2. In this step we obtain an exponential moment estimate for the left-hand side of (1.7) which will allow us to use the moment method below. More specifically, we are going to verify that for every $\eta > 0$ the expectation $\mathbb{E}^{x^{(k)}}[e^{\eta \|X^{(k)}(t)\|}]$ is bounded above by a finite constant uniformly on compact sets of $(x^{(k)}, t)$ in $\mathbb{R}^k \times [0, \infty)$. Here the notation is that of (1.1) and $\|\cdot\|$ stands for the Euclidean norm on \mathbb{R}^k .

From Itô's formula it follows that

$$\begin{aligned} d\|X^{(k)}(t)\|^2 &= \beta \sum_{\substack{1 \leq i, j \leq k \\ i \neq j}} \frac{X_i^{(k)}(t)}{X_i^{(k)}(t) - X_j^{(k)}(t)} dt + \sum_{i=1}^k 2X_i^{(k)}(t) dB_i^{(k)}(t) + k dt \\ &= \left(\beta \binom{k}{2} + k \right) dt + \sum_{i=1}^k 2X_i^{(k)}(t) dB_i^{(k)}(t). \end{aligned}$$

By computing the quadratic variation of $\|X^{(k)}\|^2$ we conclude further that $\|X^{(k)}\|^2$ is a squared Bessel process of dimension $\beta \binom{k}{2} + k$. The desired exponential moment estimate now follows from the Gaussian tail estimate for the latter (see, e.g., [RY, Section XI.1]).

Step 3. Next, we extend (3.2) from the generators $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(k+1)}$ to the semigroups $P^{(k)}(t)$, $t \geq 0$, and $P^{(k+1)}(t)$, $t \geq 0$. To this end, we apply Itô's formula to $J_\kappa(X^{(k)}(t); \theta)$ and take expectations on both sides of the resulting equation (keeping in mind the exponential moment estimate of Step 2) to obtain

$$(3.8) \quad P_k(t) J_\kappa(\cdot; \theta) = J_\kappa(\cdot; \theta) + \int_0^t P_k(s) \mathcal{A}^{(k)} J_\kappa(\cdot; \theta) ds, \quad t \geq 0,$$

for any Jack symmetric polynomial $J_\kappa(\cdot; \theta)$.

Next, we recall from (3.4) that $\mathcal{A}^{(k)} J_\kappa(\cdot; \theta)$ is given by a finite linear combination of Jack symmetric polynomials $J_\mu(\cdot; \theta)$ satisfying $\mu_i \leq \kappa_i$, $i = 1, 2, \dots, l$ (with the convention $\mu_i = 0$ whenever i exceeds the length of μ). Clearly, each $J_\mu(\cdot; \theta)$ also obeys the integral equation (3.8) and we can evaluate $\mathcal{A}^{(k)} J_\mu(\cdot; \theta)$ therein via (3.4). Iterating this procedure we end up with a system of linear ordinary integral equations, whose unique solution is given by a matrix exponential. More specifically, one can represent the action of $\mathcal{A}^{(k)}$ on the finite-dimensional vector space spanned by the Jack symmetric polynomials in consideration by a matrix, whose exponential evaluated on the vector of initial values $J_\mu(0; \theta)$ yields the solution to the system of the linear ordinary integral equations. This, and the same consideration with $(k+1)$ in place of k , show that the identity

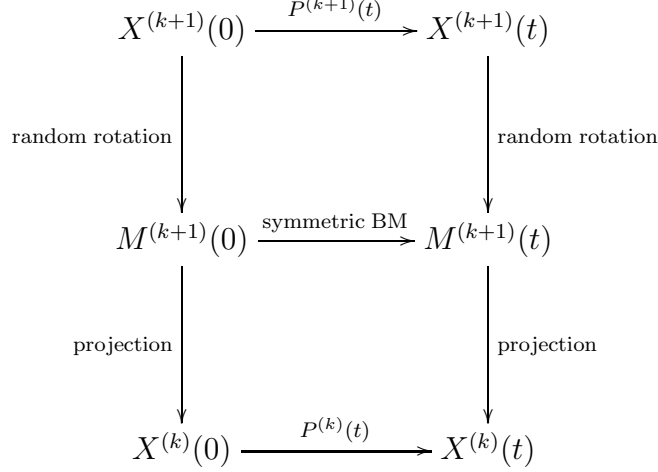
$$(3.9) \quad L^{(k)} P^{(k)}(t) J_\kappa(\cdot; \theta) = P^{(k+1)}(t) L^{(k)} J_\kappa(\cdot; \theta), \quad t \geq 0,$$

is the result of (3.2) and the following elementary lemma.

Lemma 3.1. *Suppose that three $n \times n$ matrices M_1 , M_2 , M_3 satisfy $M_1 M_2 = M_3 M_1$. Then, $M_1 e^{tM_2} = e^{tM_3} M_1$, $t \geq 0$.*

Step 4. To finish the proof of the theorem we recall from Definition 2.1 that the lexicographically leading monomial with a non-zero coefficient in $J_\kappa(\cdot; \theta)$ is $z_1^{\kappa_1} z_2^{\kappa_2} \dots z_l^{\kappa_l}$. This shows that every symmetric polynomial in k variables can be written as a finite linear combination of Jack symmetric polynomials in k variables and, thus, (3.9) extends to all symmetric polynomials. Moreover, every probability measure γ on $\mathcal{W}^{(k)}$ gives rise to a symmetrized probability measure on \mathbb{R}^k via

$$\gamma^{\text{symm}}(dz_1, dz_2, \dots, dz_k) := \frac{1}{k!} \gamma(dz_{(1)}, dz_{(2)}, \dots, dz_{(k)}),$$

FIGURE 2. Illustration of the proof of Theorem 1.1 for $\beta = 1$.

where $z_{(1)} \leq z_{(2)} \leq \dots \leq z_{(k)}$ are the order statistics of the vector (z_1, z_2, \dots, z_k) . In addition, for every polynomial p in k variables,

$$\begin{aligned} \int_{\mathbb{R}^k} p \, d\gamma^{\text{symm}} &= \int_{\mathbb{R}^k} \frac{1}{k!} \sum_{\sigma \in S_k} p(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(k)}) \, d\gamma^{\text{symm}} \\ &= \int_{\mathcal{W}^k} \frac{1}{k!} \sum_{\sigma \in S_k} p(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(k)}) \, d\gamma, \end{aligned}$$

where S_k is the set of permutations of $\{1, 2, \dots, k\}$. Using this observation for the probability measures on both sides of (1.7) we see that all moments of their symmetrized versions coincide. Therefore, in view of [dJ, Theorem 1.1 and the remark following it] applied to the left-hand side of (1.7) (recall the exponential moment estimate of Step 2), the symmetrized versions of the two sides of (1.7) must be equal. The theorem readily follows. \square

4. RANDOM MATRIX PROOF FOR $\beta = 1$

In this section we give a much simpler proof of Theorem 1.1 for the case $\beta = 1$, which relies on the random matrix interpretation of $\beta = 1$ Dyson Brownian motions. The same proof applies to the cases $\beta = 2, 4$ as well and is omitted.

Proof of Theorem 1.1 for $\beta = 1$. The strategy of the proof is to construct a process realization of the intertwining (1.7). To this end, we start with an arbitrary vector $X^{(k+1)}(0) \in \mathcal{W}^{(k+1)}$ and form the $(k+1) \times (k+1)$ diagonal matrix $D^{(k+1)}(0)$ with the components of $X^{(k+1)}(0)$ on the diagonal. Next, we sample a random $(k+1) \times (k+1)$ orthogonal matrix $O^{(k+1)}(0)$ according to the Haar measure on the orthogonal group $O(k+1)$ and introduce the symmetric matrix

$$M^{(k+1)}(0) := O^{(k+1)}(0) D^{(k+1)}(0) O^{(k+1)}(0)^{-1}.$$

We further define $X^{(k)}(0) \in \mathcal{W}^{(k)}$ as the vector of the ordered eigenvalues of the $k \times k$ top left corner of $M^{(k+1)}(0)$ (see Figure 2 for an illustration).

At this point, we recall that for a $(k+1) \times (k+1)$ random matrix from the Gaussian orthogonal ensemble (GOE, see e.g. [Fo2, Section 1.1] for more details) the eigenvalues are independent from the eigenvectors, with the matrix of eigenvectors being Haar

distributed on the orthogonal group $O(k+1)$. Hence, the conditional probability density of $X^{(k)}(0)$ given $X^{(k+1)}(0) = x^{(k+1)}$ is the same as the conditional probability density of the ordered eigenvalues of the $k \times k$ top left corner of a $(k+1) \times (k+1)$ GOE matrix given that the vector of the ordered eigenvalues of the full matrix is $x^{(k+1)}$. That conditional probability density is known to be $\Lambda^{(k)}(x^{(k+1)}, \cdot)$ (see e.g. the remark following the proof of Proposition 4.3.3 in [Fo2], as well as the proof of that proposition).

Next, we let the entries $(M^{(k+1)}(0))_{i,j}$, $1 \leq i \leq j \leq N$, of $M^{k+1}(0)$ evolve according to independent standard Brownian motions and write $M^{(k+1)}(t)$ for the completion to a symmetric matrix of the result of such an evolution. In addition, we define $X^{(k+1)}(t)$ and $X^{(k)}(t)$ to be the (random) vectors comprised of the ordered eigenvalues of $M^{(k+1)}(t)$ and the $k \times k$ top left corner of $M^{(k+1)}(t)$, respectively. Clearly, the law of $M^{(k+1)}(t)$ is invariant under conjugation by orthogonal matrices, so that its eigenvalues are independent from its eigenvectors, with the matrix of the latter being Haar distributed on the orthogonal group $O(k+1)$. As before, we conclude that the conditional probability density of $X^{(k)}(t)$ given $X^{(k+1)}(t) = x^{(k+1)}$ is $\Lambda^{(k)}(x^{(k+1)}, \cdot)$.

Finally, appealing to [AGZ, Theorem 4.3.2] (see also the original reference [Mc, p. 123]) we find that the evolutions of the processes $X^{(k+1)}$ and $X^{(k)}$ are governed by the semigroups $P^{(k+1)}(t)$, $t \geq 0$, and $P^{(k)}(t)$, $t \geq 0$, respectively. Thus, both sides of (1.7) describe the conditional distribution of $X^{(k)}(t)$ given $X^{(k+1)}(0) = x^{(k+1)}$. \square

5. INTERTWINING OF DYSON ORNSTEIN-UHLENBECK PROCESSES

In this last section we give the proof of Proposition 1.4.

Proof of Proposition 1.4. We follow the strategy of the proof of Theorem 1.1 and start by verifying

$$L^{(k)} \tilde{\mathcal{A}}^{(k)} J_{\kappa}(\cdot; \theta) = \tilde{\mathcal{A}}^{(k+1)} L^{(k)} J_{\kappa}(\cdot; \theta)$$

for all Jack symmetric polynomials $J_{\kappa}(\cdot; \theta)$. Here

$$\tilde{\mathcal{A}}^{(k)} = \mathcal{A}^{(k)} - \frac{1}{2} \mathcal{B}_3$$

is the generator of the k -dimensional Dyson Ornstein-Uhlenbeck process (see (3.1) and (2.5) for the definitions of $\mathcal{A}^{(k)}$ and \mathcal{B}_3). The identities (3.2) and (2.8) imply

$$\begin{aligned} L^{(k)} \tilde{\mathcal{A}}^{(k)} J_{\kappa}(\cdot; \theta) &= L^{(k)} \mathcal{A}^{(k)} J_{\kappa}(\cdot; \theta) - \frac{1}{2} L^{(k)} \mathcal{B}_3 J_{\kappa}(\cdot; \theta) \\ &= \mathcal{A}^{(k+1)} L^{(k)} J_{\kappa}(\cdot; \theta) - \frac{|\kappa|}{2} L^{(k)} J_{\kappa}(\cdot; \theta) = \tilde{\mathcal{A}}^{(k+1)} L^{(k)} J_{\kappa}(\cdot; \theta), \end{aligned}$$

where the last equality follows from $L^{(k)} J_{\kappa}(\cdot; \theta)$ being a multiple of $J_{\kappa}(\cdot; \theta)$ (see (2.9)) and another application of (2.8).

Next, we check that for every $\eta > 0$ the exponential moment $\mathbb{E}^{y^{(k)}}[e^{\eta \|Y^{(k)}(t)\|}]$ of the k -dimensional Dyson Ornstein-Uhlenbeck process can be bounded uniformly on

compact sets of $(y^{(k)}, t)$. To this end, we apply Itô's formula to find

$$\begin{aligned} d\|Y^{(k)}(t)\|^2 &= \beta \sum_{\substack{1 \leq i, j \leq k \\ i \neq j}} \frac{Y_i^{(k)}(t)}{Y_i^{(k)}(t) - Y_j^{(k)}(t)} dt - \|Y^{(k)}(t)\|^2 dt + \sum_{i=1}^k 2Y_i^{(k)}(t) dB_i^{(k)}(t) + k dt \\ &= \left(\beta \binom{k}{2} + k - \|Y^{(k)}(t)\|^2 \right) dt + \sum_{i=1}^k 2Y_i^{(k)}(t) dB_i^{(k)}(t). \end{aligned}$$

In other words, $\|Y^{(k)}\|^2$ solves the stochastic differential equation

$$dR(t) = \left(\beta \binom{k}{2} + k - R(t) \right) dt + 2\sqrt{R(t)} dW(t),$$

where W is a standard Brownian motion. Thus, by [KS, Proposition 5.2.18] the process $\|Y^{(k)}\|^2$ can be dominated pathwise by a squared Bessel process of dimension $\beta \binom{k}{2} + k$. The desired exponential moment estimate readily follows from the Gaussian tail estimate for the latter (see e.g. [RY, Section XI.1]).

To conclude the proof of the proposition it remains to repeat Steps 3 and 4 of the proof of Theorem 1.1 word-by-word. We omit the details. \square

REFERENCES

- [An] G. W. Anderson (1991). A short proof of Selbergs generalized beta formula. *Forum Math.* **3**, 415–417.
- [AGZ] G. W. Anderson, A. Guionnet, O. Zeitouni (2010). *An introduction to random matrices*. Cambridge Studies in Advanced Mathematics **118**. Cambridge University Press.
- [Ba] Y. Baryshnikov (2001). GUEs and queues. *Probab. Theory Related Fields* **119**, 256–274.
- [BF1] T. H. Baker, P. J. Forrester (1997). The Calogero-Sutherland model and generalized classical polynomials. *Comm. Math. Phys.* **188**, 175–216.
- [BC] A. Borodin, I. Corwin (2014). Macdonald processes. *Probab. Theory Related Fields* **158**, 225–400.
- [BF2] A. Borodin, P. L. Ferrari (2014). Anisotropic growth of random surfaces in $2+1$ dimensions. *Comm. Math. Phys.* **325**, 603–684.
- [BK] A. Borodin, J. Kuan (2010). Random surface growth with a wall and Plancherel measures for $O(\infty)$. *Comm. Pure Appl. Math.* **63**, 831–894.
- [BEY] P. Bourgade, L. Erdős, H. T. Yau (2014). Edge universality of beta ensembles. *Comm. Math. Phys.* **332**, 261–353.
- [CL1] E. Cépa, D. Lépine (1997). Diffusing particles with electrostatic repulsion. *Probab. Theory Related Fields* **107**, 429–449.
- [dJ] M. de Jeu (2003). Determinate multidimensional measures, the extended Carleman Theorem and quasi-analytic weights. *Ann. Probab.* **31**, 1205–1227.
- [DF] P. Diaconis, J. A. Fill (1990). Strong stationary times via a new form of duality. *Ann. Probab.* **18**, 1483–1522.
- [Di] A. L. Dixon (1905). Generalizations of Legendres formula $ke'(ke)k' = \frac{1}{2}\pi$. *Proc. London Math. Soc.* **3**, 206–224.
- [Dy] F. J. Dyson (1962). A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3**, 1191–1198.
- [EKYY1] L. Erdős, A. Knowles, H. T. Yau, J. Yin (2013). Spectral statistics of Erdős-Rényi graphs I: Local semicircle law. *Ann. Probab.* **41**, 2279–2375.
- [EKYY2] L. Erdős, A. Knowles, H. T. Yau, J. Yin (2012). Spectral statistics of Erdős-Rényi Graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314**, 587–640.
- [ESY] L. Erdős, B. Schlein, H. T. Yau (2011). Universality of random matrices and local relaxation flow. *Invent. Math.* **185**, 75–119.

- [Fo1] P. J. Forrester (2009). A random matrix decimation procedure relating $\beta = 2/(r + 1)$ to $\beta = 2(r + 1)$. *Comm. Math. Phys.* **285**, 653–672.
- [Fo2] P. J. Forrester (2010). *Log-gases and random matrices*. Princeton University Press.
- [GS] V. Gorin, M. Shkolnikov (2015). Multilevel Dyson Brownian motions via Jack polynomials. *Probab. Theory Related Fields* **163**, 413–463.
- [KS] I. Karatzas, S. Shreve (1991). *Brownian motion and stochastic calculus*. 2nd ed. Springer, New York.
- [MR] Z. M. Ma, M. Röckner (1992). *Introduction to the theory of (non-symmetric) Dirichlet forms*. Springer, Berlin.
- [Ma] I. G. Macdonald (1999). *Symmetric functions and Hall polynomials*. 2nd ed. Oxford University Press.
- [Mc] H. P. McKean (2005). *Stochastic integrals*. Reprint of the 1969 edition, with errata. AMS Chelsea Publishing, Providence, RI.
- [OC] N. O’Connell (2012). Directed polymers and the quantum Toda lattice. *Ann. Probab.* **40**, 437–458.
- [OO] A. Okounkov, G. Olshanski (1997). Shifted Jack polynomials, binomial formula, and applications. *Math. Res. Lett.* **4**, 69–78.
- [PS] S. Pal, M. Shkolnikov (2015). Intertwining diffusions and wave equations. Preprint available at arxiv.org/pdf/1306.0857v3.pdf.
- [RY] D. Revuz, M. Yor (1999). *Continuous martingales and Brownian motion*. 3rd ed. Springer, Berlin.
- [St] R. P. Stanley (1989). Some combinatorial properties of Jack symmetric functions. *Adv. Math.* **77**, 76–115.
- [Wa] J. Warren (2007). Dyson’s Brownian motions, intertwining and interlacing. *Electron. J. Probab.* **12**, article 19.

DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI, USA
E-mail address: kavita_ramanan@brown.edu

ORFE DEPARTMENT, PRINCETON UNIVERSITY, PRINCETON, NJ, USA
E-mail address: mshkolni@gmail.com